Study Guide for Prelim 1

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1 Methods of Integration

Substitution

Idea: Rename an ugly piece of the integrand and see if it reduces to a simpler form that you can integrate.

When? Try this method if your integral has ugly pieces like $e^{\tan v}$ or $\cos(\ln x)$, or if it looks like substitution will turn the integral into an easy one, such as $\int x^n\,dx$, $\int e^x\,dx$, or any of the other known integrals in *Table 7.1* on page 556.

Eg:

$$\text{(a)} \quad \int e^{\tan v}\sec^2 v\,dv, \quad \text{(b)} \quad \int \frac{dx}{x\cos(\ln x)}, \quad \text{(c)} \quad \int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}, \quad \text{(d)} \quad \int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}.$$

Integration By Parts

When? $\int f(x)g(x)\,dx$ where $f(x)$ can be easily differentiated, and $g(x)$ easily integrated. Good choices for $f(x)$ are $f(x) = a_0 + a_1 x + \cdots a_n x^n$ (any polynomial), $f(x) = \ln(x)$, or $f(x) = \tan^{-1}(x)$ (or any other inverse trig function). Often $g(x)$ is either $\sin(x)$, $\cos(x)$, or e^x, but $g(x)$ can be anything if $f(x)$ is a log function or an inverse trig function.

Method: Substitute $u = f(x)$ and $dv = g(x)\,dx$. Then use the formula

$$\int u\,dv = uv - \int v\,du.$$

Eg:

$$\text{(a)} \quad \int (x^2 - 5x)e^x\,dx, \quad \text{(b)} \quad \int x\ln x\,dx, \quad \text{(c)} \quad \int x(\ln x)^2\,dx.,$$

Special Case: This method sometimes works in some surprising cases. For instance, if we don't know how to integrate $f(x)$ but we do know how to differentiate it, setting $u = f(x)$ and $dv = dx$ will sometimes yield an integral we can compute.

Eg:

$$\text{(a)} \quad \int \tan^{-1}(x)\,dx, \quad \text{(b)} \quad \int \sin(\ln x)\,dx, \quad \text{(c)} \quad \int e^{\sqrt{x}}\,dx.$$

Trig Substitutions

When? $\int f(x)\,dx$ where $f(x)$ is a function with a term like $a^2 + x^2, a^2 - x^2$, or $x^2 - a^2$.

Idea: Using a trig substitution, we can replace these sums or differences of squares with a single squared trig term.

Method: Use one of the following implications to simplify the integrand:

$$\begin{aligned}
x &= a\tan\theta, \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2} \quad \Longrightarrow \quad a^2 + x^2 = a^2\sec^2\theta \\
x &= a\sin\theta, \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2} \quad \Longrightarrow \quad a^2 - x^2 = a^2\cos^2\theta \\
x &= a\sec\theta, \quad 0 \le \theta \le \frac{\pi}{2} \quad \Longrightarrow \quad x^2 - a^2 = a^2\tan^2\theta.
\end{aligned}$$

(Memorize only when to apply each substitution, then derive the implications from the basic trig identities $\sin^2\theta + \cos^2\theta = 1$, and $\sec^2\theta = 1 + \tan^2\theta$).

Eg:

$$\text{(a)} \int \frac{(1-x^2)^{3/2}}{x^6}\,dx, \quad \text{(b)} \int \frac{8x\,dx}{(4x^2+1)^2}, \quad \text{(c)} \int \frac{v^2}{(1-v^2)^{5/2}}\,dv, \quad \text{(d)} \int \frac{\sqrt{y^2-49}}{y}\,dy.$$

Algebraic Manipulation

Idea: Use algebraic manipulation to simplify the form of the integrand.

Partial Fractions

When? $\int \frac{p(x)}{q(x)}\,dx$ where $p(x)$ and $q(x)$ are polynomials, the degree of p is $<$ the degree of q and we can factor $q(x)$.

Method: The form of our new integrands will vary depending on the factorization of the denominator, however, the following case is a fairly general example. For the integrand $\frac{x+47}{(x+1)(x-1)^2(x^2+1)}$, we look for constants A, B, C, D, and E such that

$$\frac{x+47}{(x+1)(x-1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Dx+E}{x^2+1}.$$

Breaking the fraction up in this way turns our difficult integral into the sum of four simpler integrals.

Eg:

$$\text{(a)} \int \frac{x+4}{x^2+5x-6}\,dx, \quad \text{(b)} \int \frac{dx}{(x^2-1)^2},$$

$$\text{(c)} \int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds, \quad \text{(d)} \int \frac{y^4+y^2-1}{y^3+y}\,dy.$$

Improper Fractions

When? $\int \frac{p(x)}{q(x)}\,dx$ where $p(x)$ and $q(x)$ are polynomials and the degree of p is $\geq$ the degree of q.

Method: Use long division to reduce the improper fraction into the sum of a polynomial and a proper fraction of polynomials. Afterwards, the method of partial fractions is sometimes required to simplify resulting proper fraction.

Eg:

$$\text{(a)} \int \frac{x}{x+1}\,dx, \quad \text{(b)} \int \frac{2x^3}{x^2-1}\,dx, \quad \text{(c)} \int \frac{4t^3-t^2+16t}{t^2+4}\,dt.$$

Completing the Square

When? Try this method if integrand contains a quadratic of the form $q(x) = ax^2+bx+c$, where a and b are non-zero. Particularly good candidates for this method are integrals of the form $\int \frac{dx}{q(x)}$, $\int \frac{dx}{\sqrt{q(x)}}$, or $\int \frac{dx}{(x+d)\sqrt{q(x)}}$, where $q(x)$ is the quadratic.

Method: Complete the square to turn the quadratic into a square of a linear factor plus a constant; e. g. $x^2+bx+c = (x+\frac{b}{2})^2 - \frac{b^2}{4} + c$. Then use a substitution (e. g. $u = x+\frac{b}{2}$) to simplify the integrand. Sometimes a trig substition is required after the first substitution.

Eg:

$$\text{(a)} \int \frac{8\,dx}{x^2-2x+2}, \quad \text{(b)} \int \frac{dt}{\sqrt{-t^2+4t-3}}, \quad \text{(c)} \int \frac{dx}{(x+1)\sqrt{x^2+2x}}.$$

Old Tricks

Method: Apply trig identities, separate fractions, or multiply by 1 to simplify the integrand.

Eg:

$$\text{(a)} \int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta, \quad \text{(b)} \int \frac{5x+3}{\sqrt{1-x^2}}\,dx, \quad \text{(c)} \int \frac{dy}{\sqrt{e^{2y}-1}}.$$

2 Improper Integrals

Idea: A proper integral is an integral $\int_a^b f(x)\,dx$, where a and b are both finite and $f(x)$ is continuous and finite on $[a, b]$. For proper integrals, the *fundamental theorem of calculus* tells us that if F is an antiderivative of f (i.e. if F'(x) = f(x)) then $\int_a^b f(x)\,dx = F(b) - F(a)$. We will define an improper integral as a limit of proper integrals.

Comment The integral $\int_a^b f(x)\,dx$ is **improper** if: (i) a or b is infinite, or (ii) $f(x)$ is not finite on all of $[a, b]$. The fundamental theorem of calculus does not apply directly to this class of integrals. Indeed, the definition of a definite integral given in Chapter 4 explicitly assumes that the integral is proper. Without the definitions below, an improper integral would be an expression with no meaning.

Solution

Method:

1. If a is finite, and $f(x)$ is continuous on $[a, \infty]$ we define $\int_a^\infty f(x)\,dx = \lim_{c\to\infty} \int_a^c f(x)\,dx$.
2. If b is finite, and $f(x)$ is continuous on $[-\infty, b]$ we define $\int_{-\infty}^b f(x)\,dx = \lim_{c\to-\infty} \int_c^b f(x)\,dx$.
3. If a, b are finite, and $f(x)$ is continuous on $(a, b]$ we define $\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$.
4. If a, b are finite, and $f(x)$ is continuous on $[a, b)$ we b define $\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$.
5. If the integral is improper at multiple points on the domain $[a, b]$, we divide this domain into subintervals $[a, c_1], [c_1, c_2], \ldots, [c_{n-1}, c_n], [c_n, b]$ so that each subinterval contains only one problem point at one of its endpoints, and no problem points in its interior. Then define $\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^b f(x)\,dx$, applying the above rules to each piece.

Eg:

$$\text{(a)} \quad \int_0^1 x \ln x\,dx, \quad \text{(b)} \quad \int_0^\infty \frac{dx}{x^2+1}, \quad \text{(c)} \quad \int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}, \quad \text{(d)} \quad \int_{-1}^1 \frac{dx}{x^{2/3}}.$$

3 Sequences

Idea: Given a sequence of numbers $\{a_n\} = a_1, a_2, a_3, \ldots$, the question is whether or not these numbers are *converging* to some *limit* a. We say that $\{a_n\}$ converges to a, written $a_n \to a$, as $n \to \infty$, if the distance $|a_n - a|$ between the sequence points and a goes to zero as n increases.

Extend the Sequence to a Real Function

Idea: Compute the limit of a sequence as the limit of a function $f(x)$ as $x \to \infty$.

When? Use this method if the expression defining the sequence a_n can be considered as a real function in the variable n whose limit as $n \to \infty$ can be computed easily.

Method: To compute the limit of the sequence, take take the limit of its defining expression as you would take the limit of a real function. Remember L'Hopital's Rule. (The theorem behind this method is the following: If $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.

Continuous Function Theorem

Idea: It would be nice if we could move a limit inside a function, that is, if we could say

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n).$$

It is not always possible, but this theorem gives us conditions when it is.

When? Use this method if the terms of the sequence look like a continuous function applied to a much simpler sequence, for example, for sequences of the form e^{a_n}, $\ln a_n$, $\sin a_n$, or $(a_n)^p$ (constant p) where a_n converges.

Method: If a_n converges and f is continuous at $\lim_{n\to\infty} a_n$, then you can move the limit inside the function.

Eg:

$$\text{(a)}\ a_n = e^{\frac{n^4}{n^5+3n+2}},\quad \text{(b)}\ a_n = \sqrt{\left(1+\frac{1}{n}\right)^n},\quad \text{(c)}\ a_n = \tan\left(\frac{lnn}{n}\right),\quad \text{(d)}\ a_n = \left(\frac{n+1}{n}\right)^n.$$

Sandwich Theorem

Idea: If the points of our sequence $\{b_n\}$ are sandwiched between the points of two other seqenences, say $a_n \le b_n \le c_n$, and if these other sequences $\{a_n\}$ and $\{c_n\}$ both have the same limit L, then our sequence b_n must also have limit L.

When? This method is particularly useful when $a_n = b_n \cdot c_n$ where b_n is some bounded sequence and we know that $c_n \to 0$. Examples of such b_n are $b_n = 47cos^p(n)$, and $b_n = (-1)^n$.

Method: If $a_n \le b_n \le c_n$ and we know that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $b_n \to L$ as well.

Eg:

$$\text{(a)}\quad a_n = (-1)^n \frac{n^4}{\sqrt{n^9+1}},\quad \text{(b)}\quad a_n = \frac{\cos n \sin^6 n}{n^3},$$

$$\text{(c)}\quad a_n = (-1)^{2n+3} n^{1/3},\quad \text{(d)}\quad a_n = \frac{\sin(n^2+2)}{n}.$$

Algebraic Manipulation

Idea: We can often use algebraic manipulation to turn a complicated expression defining a sequence into a familiar form whose limit is known or can be computed with one of the other methods. (Important: *Know the limits of the sequences in* Table 8.1 *on page 625!!!*)

Method: Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$.

1. *(limit of the sum = sum of the limits)* $\lim_{n\to\infty}(a_n + b_n) = a + b$.
2. *(limit of the product = product of the limits)* $\lim_{n\to\infty}(a_n \cdot b_n) = ab$.
3. *(constant multiple rule)* $\lim_{n\to\infty}(k \cdot a_n) = ka$, for any constant k.
4. *(quotient rule)* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, provided $b \ne 0$.

Eg:

$$\text{(a)}\quad a_n = 1 + \frac{1}{n},\quad \text{(b)}\quad a_n = \frac{1+\sqrt[3]{n}}{47+\frac{1}{n^2}},\quad \text{(c)}\quad a_n = \left(1 - \frac{47}{n}\right)^n - \frac{22}{n}.$$

4 Infinite Series

Idea: Any finite sum $a_1 + \ldots + a_n$, can be computed by just adding up the terms. Without a definition, an infinite sum would be an expression with no meaning. We will define an infinite sum as a limit of a sequence of finite sums.

Method: Given an infinite sum $a_1 + a_2 + \ldots$, we define its kth partial sum to be $s_k = a_1 + \ldots + a_k = \sum_1^k a_n$. We then define the sum of a series to be the limit of the its *sequence of partial sums* $\{s_n\}$. If $\{s_n\} \to s$ as $n \to \infty$, we say that the infinite series $\sum_1^\infty a_n = \lim_{n\to\infty} \sum_1^n a_k = \lim_{n\to\infty} s_n$ converges and has sum s.

Computing Sums

Geometric Series

Idea: Series with terms that look like r^n for some constant r pop up everywhere. Their sums can be computed exactly with the formulas given below.

When? $\sum a_n$ where the terms of a_n are constants raised to the power n.

Method: If $|r| < 1$, the geometric series $\sum_{n=0}^\infty r^n$ and, $\sum_{n=k}^\infty ar^n$ *converge* and have sums

$$\sum_{n=0}^\infty r^n = \frac{1}{1-r}, \quad \sum_{n=k}^\infty ar^n = \frac{ar^k}{1-r}$$

On the other hand, if $|r| \geq 1$, these geometric series *diverge.*

Eg:

$$\text{(a)} \quad \sum_{n=0}^\infty \left(\frac{9}{11}\right)^n, \quad \text{(b)} \quad \sum_{n=2}^\infty \left(\frac{9}{11}\right)^{n-1}, \quad \text{(c)} \quad \sum_{n=1}^\infty \left(\frac{11}{9}\right)^n, \quad \text{(d)} \quad \sum_{n=0}^\infty 22\frac{(-2)^{2n+1}}{5^n}.$$

Telescoping Series

Idea: Sometimes cancellation simplifies the expression for the partial sum of a series into a form whose limit can evaluated easily.

When? $\sum_0^\infty a_n$ where a_n can be written $a_n = b_n - b_{n+1}$.

Method: Write out the sequence of partial sums, and find its limit directly.

Eg:

$$\text{(a)} \quad \sum_{n=1}^\infty \ln\left(\frac{n+1}{n+2}\right), \quad \text{(b)} \quad \sum_{n=0}^\infty \left(\frac{1}{n+3} + \frac{1}{n+4}\right),$$

$$\text{(c)} \quad \sum_{n=1}^\infty \frac{6}{(2n-1)(2n+1)}, \quad \text{(d)} \quad \sum_{n=1}^\infty \frac{2n+1}{n^2(n+1)^2}.$$

Algebraic Manipulation

Idea: Once again, algebraic manipulation can turn something ugly into something familar.

Method: All the algebraic tricks for simplifying integrands listed above can also be used to simplify summands. In particular, partial fractions can sometimes show that a series is telescoping. In addition, the following theorem lets us manipulate sums in the same way we manipulate integrals:

Let $\sum_0^\infty a_n = a$ and $\sum_0^\infty b_n = b$ be convergent series. Then:

1. *(sum rule)* $\sum_0^\infty (a_n + b_n) = a + b$.
2. *(constant multiple rule)* $\sum_0^\infty ka_n = ka$ for any constant k.

Eg:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{47^{2n+5}}{51^{3n}},$$

Convergence/Divergence Tests

***n*th Term Test for Divergence**

Idea: If the terms of our sequence a_n are not getting smaller, their infinite sum couldn't possibly converge.

When? Always look to see if this gives an easy answer! This is the easiest test to use on most series, and will often yield a quick divergence result.

Method: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_0^\infty a_n$ must *diverge*! (If $\lim_{n\to\infty} a_n \to 0$, this test is inconclusive.)

Eg:

$$\text{(a)} \quad \sum_{n=1}^{\infty} n \tan\frac{1}{n}, \quad \text{(b)} \quad \sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n, \quad \text{(c)} \quad \sum_{n=1}^{\infty} a_n \quad \text{where} \quad a_1 = \frac{1}{3}, a_{n+1} = \sqrt[n]{a_n}.$$

More tests...

(Several more test will be introduced before Prelim 2).