Study Guide for Prelim 2

Math 192 - Spring 1997

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1 Particularly Important Material from Prelim 1

Finding Limits of Sequences

Extend Sequence to a Real Function and Apply L'Hopital's Rule

Continuous Function Theorem

Sandwich Theorem

Algebraic Manipulation

Series

Geometric Series

Algebraic Manipulation of Series

nth Term Test for Divergence

2 Convergence Tests for Series with *Non–negative* Terms

Integral Test

Idea: If the terms of our series are *positive and decreasing*, and look like a function that we can integrate, we can compare our series with the corresponding improper integral.

When? The test applies to a series $\sum_{n=1}^{\infty} a_n$ if we can write the series in the form $\sum_{n=1}^{\infty} f(n)$ where $f(x)$ is a *positive, decreasing* function that we can integrate on the infinite interval $[1, \infty)$. This test is often more difficult to implement than others, so you might look to other tests first.

Method: If $a_n = f(n)$ (for all n), and if f a continuous, positive, decreasing function, then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

Eg:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{n \ln n}, \quad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}, \quad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}.$$

p-Series

Idea: Series of the form $\sum_0^{\infty} \frac{1}{n^p}$ where p is any constant are called *p-series.* We can always determine if a p-series converges. They are particularly useful for the comparison tests described below.

Method: By the integral test, we can show that $\sum_0^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Comment: The divergent case $p = 1$ on the boundary is called the *harmonic series.*

Eg:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{n^{3/2}}{n^2}, \quad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{47n+1}{n^3}, \quad \text{(c)} \quad \sum_{n=4}^{\infty} n^{-2}.$$

Direct Comparison Test

Idea: A series with positive terms *must* converge if its terms are all less than the corresponding terms a different series that is convergent. On the other hand, a series *must* diverge if its terms are all greater than the corresponding terms of a different series that is divergent.

When? The method will apply to any series with *non–negative* terms, however knowing when this method is best and finding the comparison series can be somewhat of an art. One clue is that the comparison series often turn out to be a p-series. Are the terms of your series all *less than* the corresponding terms of a *convergent* p-series? If not, are the terms of your series all *greater than* the corresponding terms of a *divergent* p-series? If you can easily answer Yes to either question, use the direct Comparison Test.

Method: To apply this test to a series $\sum a_n$, first guess whether or not it converges.

If you guess that it converges try to find a *convergent* series $\sum b_n$ such that (for all n), $0 \le a_n \le b_n$. If you can find one, you have proven that the original series *converges.*

If you guess that it diverges, try to find a *divergent* series $\sum b_n$ such that (for all n), $0 \le b_n \le a_n$. If you can find one you have proven that the original series *diverges.*

(Note that in both cases, the original series and the comparison series must have *non-negative* terms).

Comment: To apply this test, you must first guess whether or not your series converges. If, after you make a guess, you can't find an appropriate comparison series, there can be two reasons: either you guessed wrong, or you guessed right but (so far) you haven't been clever enough to find a comparison series that works. If you get stuck, try the alternative guess. (Also consider trying a different test).

Limit Comparison Test

Idea: Consider a series $\sum a_n$ with *non–negative* terms. It *must* converge if its terms approach 0 faster than or equally fast as do the terms of some other convergent series (with non-negative terms). On the other hand, it *must* diverge if its terms don't approach 0, or if they approach 0 equally fast as or slower than do the terms of a some other divergent series (with non–negative terms).

When? As with the Direct Comparison Test, the test will apply to any series with *non–negative terms*, but finding appropriate comparison series can be difficult. Again, one clue is that the comparison series often turn out to be a p-series. If in the limit $n \to \infty$, the terms of your series seem to behave like $\frac{1}{n^p}$, try the Limit Comparison Test with this p-series.

Method: What does it mean to say that the terms of one series approach 0 *faster than*, *equally fast as*, or *slower than* do the terms of another series? Here is a precise definition: If $a_n \to 0$ and $b_n \to 0$ then

$$a_n \to 0 \text{ faster than } b_n \to 0 \quad \textit{means} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

$$a_n \to 0 \text{ equally fast as } b_n \to 0 \quad \textit{means} \quad 0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$$

$$a_n \to 0 \text{ slower than } b_n \to 0 \quad \textit{means} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

The Limit Comparison Test works only for series with *non–negative* terms. Like with the Direct Comparison Test, to apply the test to a series $\sum a_n$ you must first guess whether or not the series converges.

If you guess that it converges try to find a *convergent* series $\sum b_n$ (with positive terms) such that $\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$. If you can find one, you have proven that the original series *converges.*

If you guess that it diverges, try to find a *divergent* series $\sum b_n$ such that $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$. If you can find one, you have proven that the original series *diverges.*

Comment: The comment made for the Direct Comparison Test can be repeated for the Limit Comparison Test: To apply the test, you must first guess whether or not your series converges. If, after you make a guess, you can't find an appropriate comparison series, there can be two reasons: either you guessed wrong, or you guessed right but (so far) you haven't been clever enough to find an appropriate comparison series. If you get stuck, try the alternative guess or consider a different method.

Eg: (For both Limit and Direct Comparison Tests)

$$\text{(a)} \sum_{n=2}^{\infty} \frac{1}{\ln n}, \quad \text{(b)} \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}, \quad \text{(c)} \sum_{n=2}^{\infty} \frac{(\ln n)^3}{n^3}, \quad \text{(d)} \sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}.$$

Ratio Test

Idea: Given an infinite series $\sum a_n$, the ratio of successive terms, a_{n+1}/a_n measures the geometric rate of decay or growth between the terms. If the terms decay geometrically in the limit, the series converges.

When? This test applies only to series $\sum a_n$ with *positive* terms. Use this test if the limit of the sequence $\{a_{n+1}/a_n\}$ can be found easily. The Ratio Test is often the best test to use if the terms a_n are a represented as a product involving factorials ($n!$).

Method: To test a series $\sum a_n$, (with *positive* terms), let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Then

1. $\sum a_n$ converges if $\rho < 1$,
2. $\sum a_n$ diverges if $\rho > 1$, but
3. the test is *inconclusive* if $\rho = 1$.

Eg:

$$\text{(a)} \sum_{n=1}^{\infty} n!e^{-n}, \quad \text{(b)} \sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}, \quad \text{(c)} \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}, \quad \text{(d)} \sum_{n=1}^{\infty} \frac{n \ln n}{2^n}.$$

Root Test

Idea: Given an infinite series $\sum a_n$, the sequence defined by $\sqrt[n]{a_n}$ also measures the geometric decay or growth of the terms of the series. Again, if the terms decay geometrically, the series converges.

When? This test applies only to series $\sum a_n$ with *non–negative* terms. Use this test if the limit of the sequence $\{\sqrt[n]{a_n}\}$ can be found easily. The Root Test is often the best test to use if the terms a_n are represented as a product whose factors are all powers of n.

Method: To test a series $\sum a_n$, (with *non–negative* terms), let $\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$. Then:

1. $\sum a_n$ converges if $\rho < 1$,
2. $\sum a_n$ diverges if $\rho > 1$, but
3. the test is *inconclusive* if $\rho = 1$.

Eg:

$$\text{(a)} \sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}, \quad \text{(b)} \sum_{n=3}^{\infty} \frac{(\ln n)^n}{n^n}, \quad \text{(c)} \sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}.$$

3 Series with Both Positive and Negative Terms

Absolute Convergence

Idea: We have useful convergence tests for series with non–negative terms. Even if the series $\sum a_n$ has both positive and negative terms, we can still apply all our old tests (ratio, root, integral, comparison, etc.) to the series $\sum |a_n|$.

Method: If $\sum |a_n|$ converges, then so does $\sum a_n$.

Caution! The divergence of $\sum |a_n|$ does not imply the divergence of $\sum a_n$.

Def. A series $\sum a_n$ is said to **converge absolutely** if its corresponding series of absolute values $\sum |a_n|$ converges.

Eg: Do the following series converge absolutely?

$$\text{(a)} \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}, \quad \text{(b)} \quad \sum_{n=1}^{\infty}(-1)^n\frac{\cos n\pi}{n\sqrt{n}},$$

$$\text{(c)} \quad \sum_{n=1}^{\infty}(-1)^n(\sqrt{n^2+n}-n), \quad \text{(d)} \quad \sum_{n=1}^{\infty}(-1)^n\frac{\tan^{-1} n}{n^2+1}.$$

Conditional Convergence

Idea: A series $\sum a_n$ might converge even if $\sum |a_n|$ does not. (Even if the magnitudes of the terms are not decreasing fast enough for $\sum |a_n|$ to converge, cancellation between the positive and negative terms of $\sum a_n$ can cause convergence).

Def. A series $\sum a_n$ is said to **converge conditionally** if it converges but its corresponding series of absolute values $\sum |a_n|$ does not.

Method: To decide if a series $\sum a_n$ converges absolutely, converges conditionally or diverges, use the following procedure: First test for absolute convergence: Does $\sum |a_n|$ converge? If so, you conclude that $\sum a_n$ converges absolutely and you're done. If not, there are two tests to distinguish between conditional convergence and divergence. The Alternating Series Test (described below) will prove convergence (which you will know is conditional convergence because you have already proven that $\sum |a_n|$ diverges). If the Alternating Series Test does not work, the only other method you know is the nth term test to prove divergence. (If neither of these methods apply you're stuck).

Alternating Series

When? $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - \cdots$ where each $u_n > 0$.

Convergence

Idea: There is an easy test for convergence of alternating series.

Method: **Alternating Series Test (Leibniz's Theorem)** The alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n$$

converges if all three of the following are true:

1. $u_n > 0$ for all n (the signs of the terms alternate).
2. $u_n \geq u_{n+1}$ for all n (the magnitudes of the terms decrease), and
3. $u_n \to 0$ as $n \to \infty$ (the terms tend to 0)

Eg: Do the following series converge absolutely, converge conditionally, or diverge?

$$\text{(a)} \quad \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}, \quad \text{(b)} \quad \sum_{n=1}^{\infty}\frac{\cos n\pi}{n}, \quad \text{(c)} \quad \sum_{n=1}^{\infty} n\cos n\pi,$$

$$\text{(d)} \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}, \quad \text{(e)} \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}.$$

Error Estimation for Alternating Series

Idea: If a series converges, we can approximate its true sum with a partial sum. For an alternating series, the following theorem estimates the error involved with this approximation.

Method: (**Alternating Series Estimation Theorem**) If all terms u_k (with $k \geq N$) of the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n$$

satisfy the conditions of Leibniz's theorem (Alternating Series Test), then for $k \geq N$,

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = (u_1 - u_2 + \cdots + (-1)^{k+1}u_k) + \text{ ERROR },$$

where $|\text{ERROR}| \leq |u_{k+1}|$.

(In other words, if we use the kth partial sum as an approximation to the true sum, then the error we make is less than the next term u_{k+1}).

Eg: For each of the following series, estimate the error involved in approximating the true sum with the sum of the first 5 terms.

$$\text{(a)} \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}, \quad \text{(b)} \quad \sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!},$$

$$\text{(c)} \quad \sum_{n=1}^{\infty}(-1)^{n}t^{n}, 0 < t < 1, \quad \text{(d)} \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.01)^n}{n}.$$

Final Comment: Series with a Few Rogue Terms in the Beginning

Idea: Many of the tests we studied have conditions such as "all terms are non–negative," or "terms decrease in magnitude," etc. You can apply these tests even if a few terms in the beginning of the series do not satisfy the conditions. All that is necessary to prove convergence or divergence is that (for some N), all terms a_k with $k \geq N$ satisfy the conditions of the test. (Note, however, that to apply the Alternating Series Estimation Theorem to a series whose first N terms don't satisfy its conditions (but the rest do), we must use a partial sum s_k with $k \geq N$ for the approximation).

Eg: Prove $\sum_{n=1}^{\infty} \frac{n-10}{n^2}$ diverges.

Solution: We can use the Limit Comparison Test (with $\sum_{n=1}^{\infty} \frac{1}{n}$ as the comparison series) even though the first 9 terms of this series are negative, because all the subsequent terms are non–negative.

4 Power Series

When? $\sum_{n=0}^{\infty} a_n(x-a)^n$

Idea: If this series converges, we can think of it as a function of the variable x. (However, it can converge for some values of x, but not for others).

Convergence

Idea: There are three possibilities: either

1. There is some $R > 0$ (the **radius of convergence**) such that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for all x with $|x-a| < R$ and diverges for $|x-a| > R$,
2. $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for all x (in which case, we say $R = \infty$), or
3. $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges at $x = a$ but diverges for all $x \neq a$ (in which case, we say $R = 0$).

Method: Test the power series for absolute convergence using the Ratio Test. (That is, test the series $\sum_{n=0}^{\infty} |a_n(x-a)^n|$ for convergence). The test will give you the inequality $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}\,|x| < 1$ as the condition for absolute convergence. Solving this inequality for x gives you the interval of absolute convergence. (And from this you can find the radius of convergence). For x outside this interval the power series diverges (because $\rho > 1$). At the two endpoints $\rho = 1$, so the test is inconclusive. You must test these endpoints separately for absolute convergence, conditional convergence or divergence.

Eg: Find the radius and interval of convergence for each of the following power series. For what values of x do they converge absolutely? converge conditionally? diverge?

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}, \quad \text{(b)} \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}, \quad \text{(c)} \quad \sum_{n=1}^{\infty} n! x^n, \quad \text{(d)} \quad \sum_{n=1}^{\infty} (-2)^n (n+1)(x-1)^n.$$

5 Taylor Series

Idea: Given a function $f(x)$, we can construct a particular power series called the **Taylor series** about some point a. The idea is that taking the first n terms of this series defines a polynomial that approximates the function well close to a.

Definitions

Def. The Taylor series for $f(x)$ about $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a.

Def. The **nth order Taylor polynomial** is sum of terms 0 through n of the Taylor series (in other words, its the nth partial sum):

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The value of polynomial at a and its first n derivatives at a agree with those of f at a.

Def. Taylor series about $x = 0$ (i.e. a is 0 in the above) is called the **Maclaurin series** for $f(x)$.

Computing Taylor and Maclaurin Series

Compute and evaluate derivatives

Idea: Use the formula given in the definition.

Method: Compute the derivatives $f(a), f'(a), f''(a), \cdots$, and use the definition

$$\text{TAYLOR SERIES} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

To find the general formula for the Taylor series (rather than just the first few terms), you will need to find the general formula for the derivatives. Try to find the pattern as you take the successive derivatives.

Eg: Find the Taylor series for the following functions about $x = a$.

$$\text{(a)} \quad e^x, \; a = 2, \quad \text{(b)} \quad x^4 + x^2 + 1, \; a = -2,$$

Algebraic Manipulation of Frequently Used Maclaurin Series

Idea: Use algebraic manipulation to compute series from other series you know (or that you're given).

Method: If $\sum a_n x^n$ is the Maclaurin Series for $f(x)$, (and c is a constant) then,

$$c\sum a_n x^n = \sum c a_n x^n \quad \text{is the Maclaurin Series for} \quad cf(x),$$
$$x\sum a_n x^n = \sum a_n x^{n+1} \quad \text{is the Maclaurin Series for} \quad xf(x),$$
$$\sum a_n (cx)^n = \sum a_n c^n x^n \quad \text{is the Maclaurin Series for} \quad f(cx).$$

In the first two cases, I brought a constant inside the sum. (The variable x is "constant" with respect to the summation index n, so I can bring it inside a sum just as easily as I can a number). In the last case, I have substituted "cx" in for "x".

These series look like power series (and indeed, they are)! When I say that $\sum a_n x^n$ is "The Maclaurin Series for f," I mean that $a_n = \frac{f^{(n)}(0)}{n!}$ (so that the expression agrees with the definition).

We can also add power series (and hence Taylor and Maclaurin Series) wherever they both converge. Thus, if $\sum b_n x^n$ is the Maclaurin Series for $g(x)$, then

$$\sum a_n x^n + \sum b_n x^n = \sum (a_n + b_n) x^n \quad \text{is the Maclaurin series for} \quad f(x) + g(x).$$

Eg: Find the Taylor series for the following functions about $x = a$.

(a) $x^2 \ln(1+3x)$, $a = 0$, (b) $\dfrac{x}{(1-x)}$, $a = 0$, (c) $e^x + e^{-x}$, $a = 0$, (d) $5\cos(\pi x), a = 0$.

Differentiation & Integration of Power Series (or Taylor or Maclaurin Series)

Idea: We can differentiate and integrate a power series (and hence a Taylor or a Maclaurin series) term–by–term inside its interval of convergence. The new series will have the same interval of convergence as the original (except that it may have different convergence properties at the endpoints of this interval).

Method: Within the interval of absolute convergence, (but not at the endpoints)

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right) = \sum_{n=0}^{\infty}\left(\frac{d}{dx} a_n (x-a)^n\right) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}.$$

Method: Within the interval of absolute convergence, (but not at the endpoints)

$$\int \left(\sum_{n=0}^{\infty} a_n (x-a)^n\right) dx = \sum_{n=0}^{\infty}\left(\int a_n (x-a)^n \, dx\right) = \sum_{n=1}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}.$$

Eg: Differentiate and integrate the following power series: What series do you get? What are their intervals of convergence?

(a) $\cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \cdots$, (b) $e^{2x} = \displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n!}$, (c) $\dfrac{1}{1-x} = \displaystyle\sum_{n=1}^{\infty} x^n$.

Frequently Used Maclaurin Series That You Should Know

Idea: The Maclaurin series for the following functions are often needed for applying the above techniques: (see page 696 and memorize them!):

$$\frac{1}{1-x}, e^x, \cos x, \sin x, \ln(1+x), (1+x)^m.$$

The series for $(1+x)^m$ is called the **binomial series**.

The series for $\ln(1+x)$ can be easily derived by integrating the series for $\frac{1}{1+x}$. The other Maclaurin series listed in the table on page 696 that I haven't listed above (that is, the series for $\frac{1}{1+x}$, $\ln(\frac{1+x}{1-x})$, and for $\tan^{-1} x$) can also all be derived easily from the others. How?

Error Estimation and Convergence of Taylor Series

Taylor's Theorem

Idea: This theorem gives us an expression for the error involved if we approximate a function with a Taylor polynomial.

Taylor's Theorem: If f and its first n derivatives are continuous at all points between x and a (including at x and at a) then there is some value c between x and a such that

$$\begin{aligned} f(x) &= P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}, \\ &= P_n(x) + R_n(x). \end{aligned}$$

Thus

$$\text{ERROR} = R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Method: Far away from a, the Taylor polynomial is no longer a good approximation to f. If you want a bound on the error that doesn't depend upon the variable x, the bound will only be valid in some interval close to a, say, $|x-a| < \delta$, (equivalently, $a - \delta \leq x \leq a + \delta$). (Sometimes the problem gives you an interval and asks you to estimate the error – that is to find a bound which the error is less than on the whole interval). Other problems specify a bound and ask you to find an interval on which the error is less than the bound).

The point c changes as x changes, but you don't know what the point c is anyway. Thus to find an explicit bound on the error using Taylor's Theorem, you must have a bound on the $n+1$st derivative. That is, you need to find a bound M such that $f^{(n+1)}(x) < M$ for *all* x in the specified interval. If the specified interval is $|x-a| < \delta$, and the bound on the nth derivative (in that interval) is $f^{(n+1)}(x) < M$, Taylor's Theorem says:

$$|\text{ERROR}| = \frac{\left|f^{(n+1)}(c)\right|}{(n+1)!}|x-a|)^{n+1} \leq \frac{M}{(n+1)!}\delta^{n+1}.$$

Convergence of Taylor Series

Idea: Does the Taylor series converge? (More precisely, does it have a positive radius of convergence?) If so, does the Taylor series converge to the original function? The answer to these questions depends on the function. The counterexample on page 677 is probably the only example of a function that you will find in Thomas and Finney for which the the Taylor series converges, but does not converge to the function.

Applying the Alternating Series Estimation Theorem to Taylor Polynomials

Idea: This theorem can also be used to estimate the error involved with approximating a function in a specified interval by a Taylor polynomial. (Or to find an interval on which the error satisfies a certain bound).

When? The same conditions must hold for a power series that do for a constant series: the signs of the terms in the Taylor Series must alternate and their magnitudes must decrease and tend to 0. Be sure that these conditions holds for all x in the specified interval. Ask yourself: What if x is negative? (You can just assume that the Taylor series converges *to the function* in its interval of convergence, even though it's not always true).

Eg: Go through problems 8.10.19-8.10.30.

Eg: Estimate the error in the following approximations for $|x| < 0.5$:

$$\text{(a)} \quad \sqrt{1+x} \sim 1 + \left(\frac{x}{2}\right), \quad \text{(b)} \quad \cos x \sim 1 - \frac{x^3}{3!}.$$